

A necessary and sufficient condition for the non-trivial limit of the derivative martingale in a branching random walk

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Summary. We consider a branching random walk on the line. Biggins and Kyprianou [6] proved that, in the boundary case, the associated derivative martingale converges almost surely to a finite nonnegative limit, whose law serves as a fixed point of a smoothing transformation (Mandelbrot's cascade). In the present paper, we give a necessary and sufficient condition for the non-triviality of this limit and establish a Kesten-Stigum-like result.

Keywords. Branching random walk; derivative martingale; Mandelbrot's cascade; random walk conditioned to stay positive.

1 Introduction

We consider a discrete-time branching random walk (BRW) on the real line, which can be described in the following way. An initial ancestor, called the root and denoted by \emptyset , is created at the origin. It gives birth to some children which form the first generation and whose positions are given by a point process \mathcal{L} on \mathbb{R} . For any integer $n \geq 1$, each individual in the n th generation gives birth independently of all others to its own children in the $(n+1)$ th generation, and the displacements of its children from this individual's position is given by an independent copy of \mathcal{L} . The system goes on if there is no extinction. We thus obtain a genealogical tree, denoted by \mathbb{T} . For each vertex (individual) $u \in \mathbb{T}$, we denote its generation by $|u|$ and its position by $V(u)$. In particular, $V(\emptyset) = 0$ and $(V(u); |u| = 1) = \mathcal{L}$.

Note that the point process \mathcal{L} plays the same role in the BRW as the offspring distribution in a Galton-Watson process. We introduce the Laplace-Stieltjes transform of \mathcal{L} as follows:

$$(1.1) \quad \Phi(t) := \mathbf{E} \left[\int_{\mathbb{R}} e^{-tx} \mathcal{L}(dx) \right] = \mathbf{E} \left[\sum_{|u|=1} e^{-tV(u)} \right], \text{ for } \forall t \in \mathbb{R}.$$

Let $\Psi(t) := \log \Phi(t)$. We always assume in this paper $\Psi(0) > 0$ so that $\mathbf{E}\left[\sum_{|u|=1} 1\right] > 1$. This yields that with strictly positive probability, the system survives. Let q be the probability of extinction. Clearly, $q < 1$.

Let $(\mathcal{F}_n; n \geq 0)$ be the natural filtration of this branching random walk, i.e. let $\mathcal{F}_n := \sigma\{(u, V(u)); |u| \leq n\}$. We introduce the additive martingale for any $t \in \mathbb{R}$,

$$(1.2) \quad W_n(t) := \sum_{|u|=n} e^{-tV(u) - n\Psi(t)}.$$

It is a nonnegative martingale with respect to $(\mathcal{F}_n; n \geq 0)$, which converges almost surely to a finite nonnegative limit. Biggins [3] established a necessary and sufficient condition for the mean convergence of $W_n(t)$, and generalized Kesten-Stigum theorem for the Galton-Watson processes. A simpler proof based on a change of measures was given later by Lyons [14].

More generally, Biggins and Kyprianou [6] studied the martingales produced by the so-called mean-harmonic functions. Given suitable conditions on the offspring distribution \mathcal{L} of the branching random walk, like the $X \log X$ condition of the Kesten-Stigum theorem, they gave a general treatment to obtain the mean convergence of these martingales. In this paper, following their ideas, we work on one special example and give a Kesten-Stigum-like theorem.

Throughout this paper, we consider the boundary case (in the sense of [7]) where $\Psi(1) = \Psi'(1) = 0$, i.e.,

$$(1.3) \quad \mathbf{E}\left[\sum_{|u|=1} e^{-V(u)}\right] = 1, \quad \mathbf{E}\left[\sum_{|u|=1} V(u)e^{-V(u)}\right] = 0.$$

In addition, we assume that

$$(1.4) \quad \sigma^2 := \mathbf{E}\left[\sum_{|u|=1} V(u)^2 e^{-V(u)}\right] \in (0, \infty).$$

We are interested in the derivative martingale, which is defined as follows:

$$(1.5) \quad D_n := \sum_{|u|=n} V(u)e^{-V(u)}, \quad \forall n \geq 0.$$

It is a signed martingale with respect to (\mathcal{F}_n) , of mean zero. By Theorem 5.1 of [6], under (1.3) and (1.4), D_n converges almost surely to a finite nonnegative limit, denoted by D_∞ . Moreover, D_∞ satisfies the following equation (Mandelbrot's cascade):

$$(1.6) \quad D_\infty = \sum_{|u|=1} e^{-V(u)} D_\infty^{(u)},$$

where $D_\infty^{(u)}$ are copies of D_∞ independent of each other and of \mathcal{F}_1 . Note that D_∞ serves as a nonnegative fixed point of a smoothing transformation. From this point of view, the questions concerning the existence, uniqueness and asymptotic behavior of such fixed points have been much studied in the literature ([5, 7, 12, 13]). We are interested in the existence of a non-trivial fixed point, and we are going to determine when $\mathbf{P}(D_\infty > 0) > 0$.

It is known that $\mathbf{P}(D_\infty = 0)$ is equal to either the extinction probability q or 1 (see [1], for example). We say that the limit D_∞ is non-trivial if $\mathbf{P}(D_\infty > 0) > 0$, which means that $\mathbf{P}(D_\infty = 0) = q$. Otherwise, it is trivially zero. In this paper, we give a sufficient and necessary condition for the non-triviality of D_∞ . The main result is stated as follows.

For any $y \in \mathbb{R}$, let $y_+ := \max\{y, 0\}$ and let $\log_+ y := \log(\max\{y, 1\})$. We introduce the following random variables:

$$(1.7) \quad Y := \sum_{|u|=1} e^{-V(u)}, \quad Z := \sum_{|u|=1} V(u)_+ e^{-V(u)}.$$

Theorem 1.1. *The limit of the derivative martingale D_n is non-trivial, namely $\mathbf{P}(D_\infty > 0) > 0$, if and only if the following condition holds:*

$$(1.8) \quad \mathbf{E}\left(Z \log_+ Z + Y(\log_+ Y)^2\right) < \infty.$$

Remark 1.2. *In [6], the authors studied the optimal condition for the non-triviality of D_∞ . However, there is a small gap between the necessary condition and the sufficient condition for $\mathbf{P}(D_\infty > 0) > 0$ in their Theorem 5.2. Our result fills this gap and gives the analogue of the result of [15] in the case of branching Brownian motion.*

Remark 1.3. *Aïdékon proved that the condition (1.8) is sufficient for $\mathbf{P}(D_\infty > 0) > 0$ (see Proposition A.3 in the Appendix of [1]).*

The paper is organized as follows. Section 2 introduces a change of measures based on a truncated martingale which is closely related to the derivative martingale. We also prove a proposition concerning certain behaviors of a centered random walk conditioned to stay positive at the end of Section 2. Then, by using this proposition, we prove Theorem 1.1 in Section 3.

Throughout the paper, $(c_i)_{i \geq 0}$ denote positive constants. We write $\mathbf{E}[f; A]$ for $\mathbf{E}[f1_A]$ and set $\sum_\emptyset := 0$.

2 Lyons' change of measures via truncated martingales

2.1 Truncated martingales

We begin with the well-known many-to-one lemma. For any $a \in \mathbb{R}$, let \mathbf{P}_a be the probability measure such that $\mathbf{P}_a\left((V(u), u \in \mathbb{T}) \in \cdot\right) = \mathbf{P}\left((V(u) + a, u \in \mathbb{T}) \in \cdot\right)$. The corresponding expectation is denoted by \mathbf{E}_a . We write \mathbf{P}, \mathbf{E} instead of $\mathbf{P}_0, \mathbf{E}_0$ for brevity. For any particle $u \in \mathbb{T}$, we denote by u_i its ancestor at the i th generation, for $0 \leq i < |u|$. In addition, we write $u_{|u|} := u$. We thus denote its ancestral line by $[\![\emptyset, u]\!] := \{u_0, u_1, \dots, u_{|u|}\}$.

Lemma 2.1 (Many-to-one). *There exists a sequence of i.i.d centered random variables $(S_{k+1} - S_k)$, $k \geq 0$ such that for any $n \geq 1$ and any measurable function $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$, we have*

$$(2.1) \quad \mathbf{E}_a \left[\sum_{|u|=n} g(V(u_1), \dots, V(u_n)) \right] = \mathbf{E}_a \left[e^{S_n - a} g(S_1, \dots, S_n) \right],$$

with $\mathbf{P}_a[S_0 = a] = 1$.

In view of (1.4), $S_1 - S_0$ has a finite variance $\sigma^2 = \mathbf{E}[S_1^2] = \mathbf{E}[\sum_{|u|=1} V(u)^2 e^{-V(u)}]$.

Let $U^-(dy)$ be the renewal measure associated with the weak descending ladder height process of $(S_n, n \geq 0)$. Following the arguments in Section 2 of [4], we obtain that for any measurable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$,

$$(2.2) \quad \mathbf{E} \left[\sum_{j=0}^{\tau-1} f(-S_j) \right] = \int_0^\infty f(y) U^-(dy),$$

where τ be the first time that (S_n) enters $(0, \infty)$, namely $\tau := \inf\{k > 0, S_k \in (0, \infty)\}$ which is proper here. We define $R(x) := U^-([0, x))$ for all $x > 0$ and define $R(0) := 1$. Note that $R(x)$ equals the renewal function $U^-([0, x])$ at points of continuity. We collect the following properties of this function $R(x)$ which are consequences of the renewal theorem (see [4, 2, 17]).

Fact 2.2. (i) *There exists a positive constant $c_0 > 0$ such that*

$$(2.3) \quad \lim_{x \rightarrow \infty} \frac{R(x)}{x} = c_0.$$

(ii) *There exist two constants $0 < c_1 < c_2 < \infty$ such that*

$$(2.4) \quad c_1(1+x) \leq R(x) \leq c_2(1+x), \quad \forall x \geq 0.$$

(iii) For any $x \geq 0$, we have $\mathbf{E}[R(S_1 + x)1_{(S_1 + x > 0)}] = R(x)$.

Let $\beta \geq 0$. Started from $V(\emptyset) = a$, we add a barrier at $-\beta$ to the branching random walk. Now, we define the following truncated random variables:

$$(2.5) \quad D_n^{(\beta)} := \sum_{|x|=n} R(V(x) + \beta)e^{-V(x)} 1_{(\min_{1 \leq k \leq n} V(x_k) > -\beta)}, \quad \forall n \geq 1,$$

$$\text{and } D_0^{(\beta)} := R(a + \beta)e^{-a} 1_{(a \geq -\beta)}.$$

Lemma 2.3. *For any $a \geq 0$ and $\beta \geq 0$, under \mathbf{P}_a , the process $(D_n^{(\beta)}, n \geq 0)$ is a nonnegative martingale with respect to $(\mathcal{F}_n, n \geq 0)$.*

This lemma follows immediately from (iii) of Fact 2.2 and the branching property. We feel free to omit its proof and call $(D_n^{(\beta)})$ the truncated martingale. It also tells us that under \mathbf{P}_a , $(D_n^{(\beta)}, n \geq 0)$ converges almost surely to a finite nonnegative limit, which we denote by $D_\infty^{(\beta)}$.

The connection between the limits of the derivative martingale and truncated martingales is recorded in the following Lemma, the proof of which can be referred to [6] and [1].

Lemma 2.4. (1) *If D_∞ is trivial, i.e., $\mathbf{P}(D_\infty = 0) = 1$, then for any $\beta \geq 0$, $D_\infty^{(\beta)}$ is trivially zero under \mathbf{P} .*

(2) *Under \mathbf{P} , if there exists some $\beta \geq 0$ such that $D_\infty^{(\beta)}$ is trivially zero, so is D_∞ .*

Thanks to Lemma 2.4, we only need to investigate the truncated martingale $(D_n^{(0)}; n \geq 0)$ and determine when its limit is non-trivial.

2.2 Lyons' change of probabilities and spinal decomposition

Let $\beta = 0$. With this nonnegative martingale $(D_n^{(0)}, n \geq 0)$, we define for any $a \geq 0$ a new probability measure \mathbf{Q}_a such that for any $n \geq 1$,

$$(2.6) \quad \frac{d\mathbf{Q}_a}{d\mathbf{P}_a} \Big|_{\mathcal{F}_n} = \frac{D_n^{(0)}}{R(a)e^{-a}}.$$

\mathbf{Q}_a is defined on $\mathcal{F}_\infty := \vee_{n \geq 0} \mathcal{F}_n$. Let us give an intuitive description of the branching random walk under \mathbf{Q}_a , which is known as the spinal decomposition. We start from one single particle ω_0 , located at the position $V(\omega_0) = a$. At time 1, it dies and produces a point process distributed as $(V(u); |u| = 1)$ under \mathbf{Q}_a . Among the children of ω_0 , ω_1 is chosen

to be u with probability proportional to $R(V(u))e^{-V(u)}1_{(V(u)>0)}$. At each time $n+1$, each particle v in the n th generation dies and produces independently a point process distributed as $(V(u); |u|=1)$ under $\mathbf{P}_{V(v)}$ except ω_n , which dies and generates independently a point process distributed as $(V(u); |u|=1)$ under $\mathbf{Q}_{V(\omega_n)}$. And then ω_{n+1} is chosen to be u among the children of ω_n , with probability proportional to $R(V(u))e^{-V(u)}1_{(\min_{1 \leq k \leq n+1} V(u_k) > 0)}$. We still use \mathbb{T} to denote the genealogical tree. Then $(\omega_n; n \geq 0)$ is an infinite ray in \mathbb{T} , which is called the spine. The rigorous proof was given in Appendix A of [1]. Indeed, this type of measures' change and the establishment of a spinal decomposition have been developed in various cases of the branching framework; see, for example [14, 11, 8, 10].

We state the following fact about the distribution of the spine process $(V(\omega_n); n \geq 0)$ under \mathbf{Q}_a .

Fact 2.5. *Let $a \geq 0$. For any $n \geq 0$ and any measurable function $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}_+$, we have*

$$(2.7) \quad \mathbf{E}_{\mathbf{Q}_a} \left[g(V(\omega_0), \dots, V(\omega_n)) \right] = \frac{1}{R(a)} \mathbf{E}_a \left[g(S_0, \dots, S_n) R(S_n); \min_{1 \leq k \leq n} S_i > 0 \right],$$

where (S_n) is the same as that in Lemma 2.1.

For convenience, let $(\zeta_n; n \geq 0)$ be a stochastic process under \mathbf{P}_a such that

$$(2.8) \quad \mathbf{P}_a \left[(\zeta_n; n \geq 0) \in \cdot \right] = \mathbf{Q}_a \left[(V(\omega_n); n \geq 0) \in \cdot \right].$$

Obviously, under \mathbf{P}_a , $(\zeta_n; n \geq 0)$ is a Markov chain with transition probabilities P so that, for any $x \geq 0$, $P(x, dy) = \frac{R(y)}{R(x)} 1_{(y>0)} \mathbf{P}_x(S_1 \in dy)$. This process (ζ_n) is usually called a random walk conditioned to stay positive. It has been arisen and studied in, for instance, [17, 2, 4, 18]. In what follows, we state some results about (ζ_n) , which will be useful later in Section 3.

2.3 Random walk conditioned to stay positive

Recall that (S_n) is a centered random walk on \mathbb{R} with finite variance σ^2 . Let τ_- be the first time that (S_n) hits $(-\infty, 0]$, namely, $\tau_- := \inf\{k \geq 1 : S_k \leq 0\}$. Let $(T_k, H_k; k \geq 0)$ be the strict ascending ladder epochs and heights of $(S_n; n \geq 0)$, i.e., $T_0 = 0$, $H_0 := S_0$ and for any $k \geq 1$, $T_k := \inf\{j > T_{k-1} : S_j > H_{k-1}\}$, $H_k := S_{T_k}$. We denote by $U(dx)$ the corresponding renewal measure (see Chapter XII in [9], for example). Then, similarly to (2.2), for any measurable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$,

$$(2.9) \quad \mathbf{E} \left[\sum_{n=0}^{\tau_- - 1} f(S_n) \right] = \mathbf{E} \left[\sum_{k \geq 0} f(H_k) \right] = \int_0^\infty f(x) U(dx).$$

We deduce from (2.7) and (2.9) that

$$\begin{aligned}
\mathbf{E}\left[\sum_{n \geq 0} f(\zeta_n)\right] &= \mathbf{E}_{\mathbf{Q}_0}\left[\sum_{n \geq 0} f(V(\omega_n))\right] = \sum_{n \geq 0} \mathbf{E}\left[f(S_n)R(S_n)1_{(\min_{1 \leq k \leq n} S_k > 0)}\right] \\
(2.10) \quad &= \mathbf{E}\left[\sum_{n=0}^{\tau_- - 1} f(S_n)R(S_n)\right] = \int_0^\infty f(x)R(x)U(\mathrm{d}x).
\end{aligned}$$

Recall also that $U^-(\mathrm{d}x)$ is the renewal measure associated with the weak descending ladder height process of (S_n) . By the renewal theorem (see P.360 in [9]), there exist two constants $c_3, c_4 > 0$ such that for $\forall x, y \geq 0$,

$$(2.11) \quad c_3(1+x) \leq U([0, x]) \leq c_4(1+x), \quad 0 \leq U([x, x+y]) \leq c_4(1+y);$$

$$(2.12) \quad c_3(1+x) \leq U^-([0, x]) \leq c_4(1+x), \quad 0 \leq U^-([x, x+y]) \leq c_4(1+y).$$

Given a non-increasing function $F \geq 0$, we present the following proposition, which gives a necessary and sufficient condition for the infinity of the series $\sum_n F(\zeta_n)$.

Proposition 2.6. *Let $F : [0, \infty) \rightarrow [0, \infty)$ be non-increasing. Then*

$$(2.13) \quad \int_0^\infty F(y)y \mathrm{d}y = \infty \iff \sum_{n \geq 0} F(\zeta_n) = \infty, \quad \mathbf{P}\text{-a.s.}$$

Note that (ζ_n) can be viewed as a discrete-time counterpart of the 3-dimensional Bessel process, for which a similar result holds (see, for instance, Ex 2.5, Chapter XI of [16]). And we will prove (2.13) in a similar way as for the Bessel process.

Proof. Observe that $0 \leq F(x) \leq F(0) < \infty$ for any $x \geq 0$. So there is no difference between the two events $\{\sum_{n \geq 0} F(\zeta_n) = \infty\}$ and $\{\sum_{n \geq 1} F(\zeta_n) = \infty\}$.

We first prove “ \Leftarrow ” in (2.13). It follows from (2.4) and (2.11) that

$$(2.14) \quad \int_0^\infty F(y)y \mathrm{d}y = \infty \iff \int_0^\infty F(y)R(y)U(\mathrm{d}y) = \infty.$$

Actually, by (2.10),

$$\mathbf{E}\left[\sum_{n \geq 0} F(\zeta_n)\right] = \int_0^\infty F(y)R(y)U(\mathrm{d}y).$$

Clearly, $\mathbf{P}\left[\sum_{n \geq 0} F(\zeta_n) = \infty\right] = 1$ yields $\int_0^\infty F(y)R(y)U(\mathrm{d}y) = \infty$. The “ \Leftarrow ” in (2.13) is hence proved.

To prove “ \implies ” in (2.13), we only need to show that if $\mathbf{P}\left[\sum_{n \geq 0} F(\zeta_n) = \infty\right] < 1$, then $\int_0^\infty F(y)y dy < \infty$. From now on, we suppose that $\mathbf{P}\left[\sum_{n \geq 0} F(\zeta_n) = \infty\right] < 1$, which is equivalent to say that,

$$(2.15) \quad \mathbf{P}\left[\sum_{n \geq 1} F(\zeta_n) < \infty\right] > 0.$$

We draw support from Tanaka’s construction for the random walk conditioned to stay positive ([17, 4]). Recall that $\tau = \inf\{k \geq 1 : S_k \in (0, \infty)\}$. We hence obtain an excursion $(S_j; 0 \leq j \leq \tau)$, which is denoted by $\xi = (\xi(j), 0 \leq j \leq \tau)$. Let $\{\xi_k = (\xi_k(j), 0 \leq j \leq \tau_k); k \geq 1\}$ be a sequence of independent copies of ξ . For any $k \geq 1$, let

$$(2.16) \quad \nu_k(j) := \xi_k(\tau_k) - \xi_k(\tau_k - j), \quad \forall 0 \leq j \leq \tau_k.$$

This brings out another sequence of i.i.d. excursions $\{\nu_k = (\nu_k(j), 0 \leq j \leq \tau_k); k \geq 1\}$, based on which we reconstruct the random walk conditioned to stay position (ζ_n) in the following way. Define for any $k \geq 1$,

$$(2.17) \quad T_k^+ := \tau_1 + \dots + \tau_k;$$

$$(2.18) \quad H_k^+ := \nu_1(\tau_1) + \dots + \nu_k(\tau_k) = \xi_1(\tau_1) + \dots + \xi_k(\tau_k),$$

and let $T_0^+ = H_0^+ = 0$. Then the process

$$(2.19) \quad \zeta_n = H_k^+ + \nu_{k+1}(n - T_k^+), \quad \text{for } T_k^+ < n \leq T_{k+1}^+,$$

with $\zeta_0 = 0$, is what we need.

We actually establish un process distributed as (ζ_n) . For brevity, we still denote it by (ζ_n) without changing any conclusion in this proof. For any $k \geq 1$, let

$$(2.20) \quad \chi_k(F) := \sum_{n=T_{k-1}^++1}^{T_k^+} F(\zeta_n) = \sum_{j=1}^{\tau_k} F\left(H_{k-1}^+ + \nu_k(j)\right),$$

so that $\sum_{n \geq 1} F(\zeta_n) = \sum_{k \geq 1} \chi_k(F)$.

By (2.16), we get that

$$\begin{aligned} \chi_k(F) &= \sum_{j=1}^{\tau_k} F\left(H_{k-1}^+ + \xi_k(\tau_k) - \xi_k(\tau_k - j)\right) \\ &= \sum_{j=0}^{\tau_k-1} F\left(H_k^+ - \xi_k(j)\right). \end{aligned}$$

(2.15) hence becomes that

$$(2.21) \quad \mathbf{P}\left[\sum_{k \geq 1} \chi_k(F) < \infty\right] = \mathbf{P}\left[\sum_{k \geq 1} \sum_{j=0}^{\tau_k-1} F(H_k^+ - \xi_k(j)) < \infty\right] > 0.$$

By Theorem 1 in Chapter XVIII.5 of [9], as (S_n) is of finite variance, we have $b^+ := \mathbf{E}[H_1^+] = \mathbf{E}[S_\tau] < \infty$. It follows from Strong Law of Large Numbers that \mathbf{P} -a.s.,

$$(2.22) \quad \lim_{k \rightarrow \infty} \frac{H_k^+}{k} = b^+.$$

Let $A > \max\{1, b^+\}$. This tells us that \mathbf{P} -a.s., for all large k , $H_k^+ \leq Ak$. As F is non-increasing, one sees that

$$(2.23) \quad \mathbf{P}\left[\sum_{k \geq 1} \sum_{j=0}^{\tau_k-1} F(Ak - \xi_k(j)) < \infty\right] \geq \mathbf{P}\left[\sum_{k \geq 1} \sum_{j=0}^{\tau_k-1} F(H_k^+ - \xi_k(j)) < \infty\right] > 0.$$

For any $k \geq 1$ let

$$(2.24) \quad \tilde{\chi}_k := \sum_{j=0}^{\tau_k-1} F(Ak - \xi_k(j)).$$

So, $\mathbf{P}\left[\sum_{k \geq 1} \tilde{\chi}_k < \infty\right] > 0$. Recall that $\{\xi_k, k \geq 1\}$ is a sequence of independent copies of $(S_j; 0 \leq j \leq \tau)$. This yields the independence of the sequence $\{\tilde{\chi}_k, k \geq 1\}$. It follows from Kolmogorov's 0-1 law that

$$(2.25) \quad \mathbf{P}\left[\sum_{k \geq 1} \sum_{j=0}^{\tau_k-1} F(Ak - \xi_k(j)) < \infty\right] = \mathbf{P}\left[\sum_{k \geq 1} \tilde{\chi}_k < \infty\right] = 1.$$

Moreover, let $E_M := \left\{\sum_{k \geq 1} \tilde{\chi}_k < M\right\}$ for any $M > 0$. Either there exists some $M_0 < \infty$ such that $\mathbf{P}[E_{M_0}] = 1$, or $\mathbf{P}[E_M] < 1$ for all $M \in (0, \infty)$. On the one hand, if $\mathbf{P}[E_{M_0}] = 1$ for some $M_0 < \infty$, then

$$\begin{aligned} M_0 &\geq \mathbf{E}\left[\sum_{k \geq 1} \tilde{\chi}_k\right] = \mathbf{E}\left[\sum_{k \geq 1} \sum_{j=0}^{\tau_k-1} F(Ak - \xi_k(j))\right] \\ &= \sum_{k \geq 1} \mathbf{E}\left[\sum_{j=0}^{\tau-1} F(Ak - S_j)\right] \\ &= \sum_{k \geq 1} \int_0^\infty F(Ak + y) U^-(dy), \end{aligned}$$

where the last equality follows from (2.9). One sees that $\sum_{k \geq 1} \int_0^\infty F(Ak+y)U^-(dy) < \infty$. It follows from the renewal theorem that there exists $B > 0$ such that $U^-([jB, jB+B)) > \delta > 0$ for any $j \geq 0$. As F is non-increasing,

$$(2.26) \quad \sum_{k \geq 1} \sum_{j \geq 1} F(Ak + Bj)\delta \leq \sum_{k \geq 1} \int_0^\infty F(Ak + y)U^-(dy) < \infty.$$

We hence observe that $\int_A^\infty dz \int_B^\infty F(y+z)dy \leq \sum_{k \geq 1} \sum_{j \geq 1} F(Ak + Bj)AB < \infty$. This implies that

$$\int_0^\infty F(x)x dx = \int_0^\infty dz \int_0^\infty F(z+y)dy \leq F(0)AB + \int_A^\infty dz \int_B^\infty F(y+z)dy < \infty,$$

which is what we need.

On the other hand, if $\mathbf{P}[E_M] < 1$ for all $M \in (0, \infty)$, we have $\lim_{M \uparrow \infty} \mathbf{P}[E_M] = 1$ because of (2.25). For any $k \geq 1$ and any $\ell \geq 1$, define:

$$(2.27) \quad \Lambda_\ell^{(k)} := \sum_{j=0}^{\tau_k-1} 1_{\{A(\ell-1) \leq -\xi_k(j) < A\ell\}}.$$

As $\sum_{\ell \geq 1} 1_{\{A(\ell-1) \leq -\xi_k(j) < A\ell\}} = 1$, we get that for any $k \geq 1$,

$$\begin{aligned} \tilde{\chi}_k &= \sum_{j=0}^{\tau_k-1} F(Ak - \xi_k(j)) \sum_{\ell \geq 1} 1_{\{A(\ell-1) \leq -\xi_k(j) < A\ell\}} \\ &= \sum_{\ell \geq 1} \sum_{j=0}^{\tau_k-1} F(Ak - \xi_k(j)) 1_{\{A(\ell-1) \leq -\xi_k(j) < A\ell\}} \\ &\geq \sum_{\ell \geq 1} F(Ak + A\ell) \Lambda_\ell^{(k)}, \end{aligned}$$

where the last inequality holds because F is non-increasing. It follows that

$$\begin{aligned} \sum_{k \geq 1} \tilde{\chi}_k &\geq \sum_{k \geq 1} \sum_{\ell \geq 1} F(Ak + A\ell) \Lambda_\ell^{(k)} = \sum_{n=2}^\infty F(An) \sum_{k=1}^{n-1} \Lambda_{n-k}^{(k)} \\ (2.28) \quad &= \sum_{m=1}^\infty F(Am + A)mY_m, \end{aligned}$$

where

$$(2.29) \quad Y_m := \frac{\sum_{k=1}^m \Lambda_{m+1-k}^{(k)}}{m}, \quad \forall m \geq 1.$$

We claim that there exists a $M_1 > 0$ sufficiently large such that for any $m \geq 1$,

$$(2.30) \quad c_6 \geq \mathbf{E}[Y_m 1_{E_{M_1}}] \geq c_5 > 0,$$

where c_5, c_6 are positive constants. We postpone the proof of (2.30) and go back to (2.28). It follows that

$$(2.31) \quad \begin{aligned} M_1 &\geq \mathbf{E}\left[1_{E_{M_1}} \sum_{k \geq 1} \tilde{\chi}_k\right] \geq \mathbf{E}\left[1_{E_{M_1}} \sum_{m=1}^{\infty} F(Am + A)mY_m\right] \\ &\geq \sum_{m \geq 1} F(Am + A)m\mathbf{E}[Y_m 1_{E_{M_1}}]. \end{aligned}$$

By (2.30), we obtain that

$$(2.32) \quad \sum_{m \geq 1} F(Am + A)m \leq M_1/c_5 < \infty.$$

This implies that $\int_0^\infty F(y)y dy < \infty$ thus completes the proof of Proposition 2.6.

It remains to prove (2.30).

We begin with the first and second moments of Y_m . Since $\{\omega_k; k \geq 1\}$ are i.i.d. copies of $(S_j, 0 \leq j \leq \tau)$, $(\Lambda_\ell^{(k)}; \ell \geq 1), k \geq 1$ are i.i.d. This yields that

$$(2.33) \quad \begin{aligned} \mathbf{E}[Y_m] &= \frac{1}{m} \sum_{k=1}^m \mathbf{E}[\Lambda_{m+1-k}^{(k)}] = \frac{1}{m} \sum_{k=1}^m \mathbf{E}[\Lambda_{m+1-k}^{(1)}] \\ &= \frac{1}{m} \mathbf{E}\left[\sum_{k=1}^m \Lambda_k^{(1)}\right] = \frac{1}{m} \mathbf{E}\left[\sum_{j=0}^{\tau-1} 1_{\{-S_j < Am\}}\right] \\ &= \frac{R(Am)}{m}. \end{aligned}$$

where the last equality comes from (2.2). By (2.4), for any $m \geq 1$,

$$(2.34) \quad c_1 A \leq \mathbf{E}[Y_m] \leq c_2(A + 1) =: c_6.$$

Obviously, we have $\mathbf{E}[Y_m 1_{E_M}] \leq c_6$ for any $m \geq 1$ and any $M > 0$. The fact that $\Lambda_k^{(k)}, k \geq 1$, are i.i.d. yields also that

$$(2.35) \quad \text{Var}(Y_m) = \frac{1}{m^2} \sum_{k=1}^m \text{Var}(\Lambda_k^{(1)}) \leq \frac{1}{m^2} \sum_{k=1}^m \mathbf{E}[(\Lambda_k^{(1)})^2].$$

Note that $\Lambda_1^{(1)}$ is distributed as $\sum_{j=0}^{\tau-1} 1_{\{-S_j < A\}}$ with $\tau = \inf\{k > 0 : S_k > 0\}$. We see that

$$\begin{aligned} \mathbf{E}\left[\left(\Lambda_1^{(1)}\right)^2\right] &= \mathbf{E}\left[\left(\sum_{j=0}^{\tau-1} 1_{\{-S_j < A\}}\right)^2\right] \\ &\leq 2\mathbf{E}\left[\sum_{j=0}^{\tau-1} 1_{\{-S_j < A\}} \sum_{k=j}^{\tau-1} 1_{\{-S_k < A\}}\right]. \end{aligned}$$

By Markov property, we obtain that

$$(2.36) \quad \mathbf{E}\left[\left(\Lambda_1^{(1)}\right)^2\right] \leq 2\mathbf{E}\left[\sum_{j=0}^{\tau-1} 1_{\{-S_j < A\}} R(A, -S_j)\right],$$

where

$$(2.37) \quad R(x, y) := \mathbf{E}\left[\sum_{i=0}^{\tau_y-1} 1_{\{S_i > y-x\}}\right] \text{ with } \tau_y := \inf\{k > 0 : S_k > y\} \text{ for } x, y \geq 0.$$

It follows from (2.2) that

$$(2.38) \quad \mathbf{E}\left[\left(\Lambda_1^{(1)}\right)^2\right] \leq 2 \int_0^A R(A, y) U^-(dy).$$

Consider now the strict ascending ladder epochs and heights (T_k, H_k) of (S_n) . We get that

$$R(x, y) = \mathbf{E}\left[\sum_{k=0}^{\infty} 1_{\{y \geq H_k > y-x\}} \sum_{n=T_k}^{T_{k+1}-1} 1_{\{S_n > y-x\}}\right].$$

By applying the Markov property at the times $(T_k; k \geq 1)$ and (2.2), we have for $x, y \geq 0$,

$$(2.39) \quad R(x, y) = \mathbf{E}\left[\sum_{k \geq 0} R(H_k + x - y) 1_{\{y \geq H_k > y-x\}}\right] = \int_{(y-x)_+}^y R(x - y + z) U(dz).$$

Plugging it into (2.38) then using (2.4), (2.12) and (2.11) implies that

$$(2.40) \quad \mathbf{E}\left[\left(\Lambda_1^{(1)}\right)^2\right] \leq c_7(1+A)^3 \leq c_8 A^3,$$

(see also Lemma 2 in [4]).

Moreover, for any $\ell \geq 2$, $\Lambda_\ell^{(1)}$ has the same law as $\sum_{j=0}^{\tau-1} 1_{\{\ell A - A \leq -S_j < \ell A\}}$. Similarly, we get that

$$\begin{aligned} \mathbf{E}\left[\left(\Lambda_\ell^{(1)}\right)^2\right] &= \mathbf{E}\left[\left(\sum_{j=0}^{\tau-1} 1_{\{\ell A - A \leq -S_j < \ell A\}}\right)^2\right] \\ &\leq 2\mathbf{E}\left[\sum_{j=0}^{\tau-1} 1_{\{\ell A - A \leq -S_j < \ell A\}} \sum_{k=j}^{\tau-1} 1_{\{\ell A - A \leq -S_k < \ell A\}}\right]. \end{aligned}$$

Once again, by Markov property then by (2.2),

$$\begin{aligned}\mathbf{E}\left[\left(\Lambda_\ell^{(1)}\right)^2\right] &\leq 2\mathbf{E}\left[\sum_{j=0}^{\tau-1} 1_{\{\ell A - A \leq -S_j < \ell A\}} \left(R(\ell A, -S_j) - R(\ell A - A, -S_j)\right)\right] \\ &= 2 \int_{\ell A - A}^{\ell A} \left(R(\ell A, y) - R(\ell A - A, y)\right) U^-(dy).\end{aligned}$$

Plugging (2.39) into it yields that for $\ell \geq 2$,

$$\begin{aligned}\mathbf{E}\left[\left(\Lambda_\ell^{(1)}\right)^2\right] &\leq 2 \int_{\ell A - A}^{\ell A} \left(\int_0^y R(\ell A - y + z) U(dz) - \int_{y - \ell A + A}^y R(\ell A - A - y + z) U(dz)\right) U^-(dy) \\ &= 2 \int_{\ell A - A}^{\ell A} \left(\int_0^{y - \ell A + A} R(\ell A - y + z) U(dz) \right. \\ &\quad \left. + \int_{y - \ell A + A}^y U^-([\ell A - A - y + z, \ell A - y + z]) U(dz)\right) U^-(dy),\end{aligned}$$

where the last equality holds because $R(x) = U^-([0, x])$. Observe that $R(\ell A - y + z) \leq R(A)$ for $0 \leq z \leq y - \ell A + A$ and $\ell A - A \leq y \leq \ell A$. Recall that $A \geq 1$. By (2.4), (2.11) and (2.12),

$$\begin{aligned}\mathbf{E}\left[\left(\Lambda_\ell^{(1)}\right)^2\right] &\leq c_9 \int_{\ell A - A}^{\ell A} \left(\int_0^{y - \ell A + A} (A + 1) U(dz) + \int_{y - \ell A + A}^y (1 + A) U(dz)\right) U^-(dy) \\ &\leq c_{10} (A + 1) \int_{\ell A - A}^{\ell A} (y + 1) U^-(dy) \\ &\leq c_{11} \ell A^3,\end{aligned}$$

with $c_{11} \geq c_8$. Going back to (2.35), for any $m \geq 1$,

$$(2.41) \quad \text{Var}(Y_m) \leq \frac{\sum_{\ell=1}^m c_{11} \ell A^3}{m^2} \leq c_{12} A^3.$$

Combining this with (2.34) implies that $\mathbf{E}[Y_m^2] = \text{Var}(Y_m) + \mathbf{E}[Y_m]^2 \leq c_2^2(1 + A)^2 + c_{12} A^3$.

We then use Paley-Zygmund inequality to obtain that

$$(2.42) \quad \mathbf{P}\left[Y_m > \frac{1}{2} \mathbf{E}[Y_m]\right] \geq \frac{\mathbf{E}[Y_m]^2}{4 \mathbf{E}[Y_m^2]} \geq \frac{c_1^2 A^2}{4(c_2^2(1 + A)^2 + c_{12} A^3)} := c_{13} > 0.$$

So for any $0 \leq u \leq c_1 A/2 \leq \mathbf{E}[Y_m]/2$, we have

$$(2.43) \quad \mathbf{P}(Y_m \leq u) \leq \mathbf{P}(Y_m \leq \mathbf{E}[Y_m]/2) \leq 1 - c_{13}.$$

There exists $M_1 > 0$ such that $\mathbf{P}(E_{M_1}) \geq 1 - c_{13}/2$, since $\lim_{M \uparrow \infty} \mathbf{P}[E_M] = 1$. For such $M_1 > 0$,

$$(2.44) \quad \mathbf{E}[Y_m 1_{E_{M_1}}] = \mathbf{E}\left[\int_0^{Y_m} 1_{E_{M_1}} du\right] = \int_0^\infty \mathbf{P}[\{Y_m > u\} \cap E_{M_1}] du.$$

Notice that $\mathbf{P}[\{Y_m > u\} \cap E_{M_1}] \geq \left(\mathbf{P}[E_{M_1}] - \mathbf{P}[Y_m \leq u]\right)_+$, which is larger than $c_{13}/2$ when $0 \leq u \leq c_1 A/2$. As a consequence,

$$(2.45) \quad \mathbf{E}[Y_m 1_{E_{M_1}}] = \int_0^\infty \mathbf{P}[\{Y_m > u\} \cap E_{M_1}] du \geq \int_0^{c_1 A/2} \frac{c_{13}}{2} du = \frac{c_1 c_{13} A}{4} =: c_5 > 0.$$

This completes the proof of (2.30), hence completes the proof of “ \implies ” in (2.13). Proposition 2.6 is proved. \square

3 Proof of the main theorem

Recall that we are in the regime that

$$(3.1) \quad \mathbf{E}\left[\sum_{|u|=1} e^{-V(u)}\right] = 1, \quad \mathbf{E}\left[\sum_{|u|=1} V(u) e^{-V(u)}\right] = 0, \quad \sigma^2 = \mathbf{E}\left[\sum_{|u|=1} V(u)^2 e^{-V(u)}\right] < \infty.$$

Recall also that equivalence in Theorem 1.1 is as follows:

$$(3.2) \quad \mathbf{E}\left[Y \left(\log_+ Y\right)^2\right] + \mathbf{E}\left[Z \log_+ Z\right] < \infty \iff \mathbf{P}[D_\infty > 0] > 0.$$

with $Y = \sum_{|u|=1} e^{-V(u)}$ and $Z = \sum_{|u|=1} V(u) e^{-V(u)}$.

This section is devoted to proving that the condition on the left-hand side of (3.2) (i.e. (1.8)) is necessary and sufficient for mean convergence of the truncated martingale $\left\{D_n^{(0)} = \sum_{|u|=n} R(V(u)) e^{-V(u)} 1_{\{V(u_k) > 0, \forall 1 \leq k \leq n\}}; n \geq 0\right\}$. In view of Lemma 2.4, this follows the non-triviality of D_∞ , hence proves Theorem 1.1.

In what follows, we state a result about the mean convergence of the truncated martingale $\left\{D_n^{(0)}; n \geq 0\right\}$, which is one special case of Theorem 2.1 in Biggins and Kyprianou [6].

Define

$$(3.3) \quad X := \frac{D_1^{(0)}}{D_0^{(0)}} 1_{(D_0^{(0)} > 0)} + 1_{(D_0^{(0)} = 0)}.$$

Then for any $a \geq 0$, under \mathbf{P}_a , $X = \frac{\sum_{|u|=1} R(V(u)) e^{-V(u)} 1_{(V(u) > 0)}}{R(a) e^{-a}}$.

Theorem 3.1 (Biggins and Kyprianou [6]). (ζ_n) is a random walk conditioned to stay positive, whose law was given in (2.8).

(i) If

$$(3.4) \quad \mathbf{P}\text{-a.s.} \quad \sum_{n \geq 1} \mathbf{E}_{\zeta_n} \left[X \left(R(\zeta_n) e^{-\zeta_n} X \wedge 1 \right) \right] < \infty,$$

then $\mathbf{E}[D_\infty^{(0)}] = R(0)$.

(ii) If for all $y > 0$,

$$(3.5) \quad \mathbf{P}\text{-a.s.} \quad \sum_{n=1}^{\infty} \mathbf{E}_{\zeta_n} \left[X; R(\zeta_n) e^{-\zeta_n} X \geq y \right] = \infty,$$

then $\mathbf{E}[D_\infty^{(0)}] = 0$.

Our proof relies on this theorem. First, in Subsection 3.1, we give a short proof for the sufficient part to accomplish our arguments even though it has already been proved in [1]. In Subsection 3.2, we prove that (1.8) is also the necessary condition by using Proposition 2.6.

3.1 (1.8) is a sufficient condition

This subsection is devoted to proving that

$$(3.6) \quad \mathbf{E} \left[Y \left(\log_+ Y \right)^2 \right] + \mathbf{E} \left[Z \log_+ Z \right] < \infty \implies \mathbf{E}[D_\infty^{(0)}] = R(0) = 1.$$

Proof of (3.6). According to (i) of Theorem 3.1, it suffices to show that

$$(3.7) \quad \mathbf{E} \left[Y \left(\log_+ Y \right)^2 \right] + \mathbf{E} \left[Z \log_+ Z \right] < \infty \implies \mathbf{P}\text{-a.s.} \quad \sum_{n \geq 0} \mathbf{E}_{\zeta_n} \left[X \left(R(\zeta_n) e^{-\zeta_n} X \wedge 1 \right) \right] < \infty.$$

For any particle $x \in \mathbb{T} \setminus \{\emptyset\}$, we denote its parent by \overleftarrow{u} and define its relative displacement by

$$(3.8) \quad \Delta V(u) := V(u) - V(\overleftarrow{u}).$$

Then for any $a \in \mathbb{R}$, under \mathbf{P}_a , $(\Delta V(u); |u| = 1)$ is distributed as \mathcal{L} . Let $\tilde{Y} := \sum_{|u|=1} e^{-\Delta V(u)}$ and $\tilde{Z} := \sum_{|u|=1} \left(\Delta V(u) \right)_+ e^{-\Delta V(u)}$ so that $\mathbf{P}_a \left[\left(\tilde{Y}, \tilde{Z} \right) \in \cdot \right] = \mathbf{P}[(Y, Z) \in \cdot]$.

Note that under \mathbf{P}_{ζ_n} ,

$$\begin{aligned}
X &= \frac{\sum_{|u|=1} R(V(u))e^{-V(u)}1_{(V(u)>0)}}{R(\zeta_n)e^{-\zeta_n}} \\
(3.9) \quad &= \frac{\sum_{|u|=1} R(\zeta_n + \Delta V(u))e^{-\Delta V(u)}1_{(\Delta V(u)>-\zeta_n)}}{R(\zeta_n)},
\end{aligned}$$

where $(\Delta V(u); |u| = 1)$ is independent of ζ_n . By (2.4), it follows that

$$\begin{aligned}
X &\leq \frac{\sum_{|u|=1} c_2(\zeta_n + 1)e^{-\Delta V(u)}1_{(\Delta V(u)>-\zeta_n)}}{R(\zeta_n)} + \frac{\sum_{|u|=1} c_2\Delta V(u)e^{-\Delta V(u)}1_{(\Delta V(u)>-\zeta_n)}}{R(\zeta_n)} \\
&\leq \frac{c_2}{c_1} \sum_{|u|=1} e^{-\Delta V(u)} + c_2 \frac{\sum_{|u|=1} \Delta V(u)_+ e^{-\Delta V(u)}}{R(\zeta_n)} \\
&\leq c_{14} \left(\tilde{Y} + \frac{\tilde{Z}}{R(\zeta_n)} \right) \leq 2c_{14} \max \left\{ \tilde{Y}, \frac{\tilde{Z}}{R(\zeta_n)} \right\},
\end{aligned}$$

where (\tilde{Y}, \tilde{Z}) is independent of ζ_n . This implies that

$$\begin{aligned}
&\sum_{n \geq 1} \mathbf{E}_{\zeta_n} \left[X \left(R(\zeta_n) e^{-\zeta_n} X \wedge 1 \right) \right] \\
&\leq c_{15} \left(\sum_{n \geq 0} \mathbf{E} \left[\tilde{Y} \left(R(\zeta_n) e^{-\zeta_n} \tilde{Y} \wedge 1 \right) \middle| \zeta_n \right] + \sum_{n \geq 0} \frac{1}{R(\zeta_n)} \mathbf{E} \left[\tilde{Z} \left(e^{-\zeta_n} \tilde{Z} \wedge 1 \right) \middle| \zeta_n \right] \right) \\
(3.10) \quad &=: c_{15} (\Sigma_1 + \Sigma_2).
\end{aligned}$$

Hence we only need to prove that

$$(3.11) \quad \mathbf{E} \left[Y \left(\log_+ Y \right)^2 \right] + \mathbf{E} \left[Z \log_+ Z \right] < \infty \implies \mathbf{E} [\Sigma_1] + \mathbf{E} [\Sigma_2] < \infty,$$

which leads to (3.7). On the one hand, as (2.4) gives that $R(x) \leq c_{16}e^{x/2}$ for all $x \geq 0$, we see that

$$\begin{aligned}
\mathbf{E} [\Sigma_1] &\leq c_{17} \mathbf{E} \left[\sum_{n \geq 0} \mathbf{E} \left[\tilde{Y} \left(e^{-\zeta_n/2} \tilde{Y} \wedge 1 \right) \middle| \zeta_n \right] \right] \\
&= c_{17} \sum_{n \geq 0} \mathbf{E} \left[\left(\tilde{Y} \right)^2 e^{-\zeta_n} 1_{\{\tilde{Y} \leq e^{\zeta_n/2}\}} + \tilde{Y} 1_{\{\tilde{Y} > e^{\zeta_n/2}\}} \right] \\
&= c_{17} \mathbf{E} \left\{ \left(\tilde{Y} \right)^2 \mathbf{E} \left[\sum_{n \geq 0} e^{-\zeta_n} 1_{\{\zeta_n \geq 2 \log \tilde{Y}\}} \middle| \tilde{Y} \right] + \tilde{Y} \mathbf{E} \left[\sum_{n \geq 0} 1_{\{\zeta_n < 2 \log \tilde{Y}\}} \middle| \tilde{Y} \right] \right\},
\end{aligned}$$

where \tilde{Y} and (ζ_n) are independent. By (2.10),

$$(3.12) \quad \mathbf{E}[\Sigma_1] \leq c_{17} \mathbf{E} \left[\left(\tilde{Y} \right)^2 \int_{2 \log_+ \tilde{Y}}^{\infty} e^{-x} R(x) U(\mathrm{d}x) + \tilde{Y} \int_0^{2 \log_+ \tilde{Y}} R(x) U(\mathrm{d}x) \right],$$

which by (2.4) and (2.11) implies that

$$(3.13) \quad \mathbf{E}[\Sigma_1] \leq c_{17} c_2 \mathbf{E} \left[\left(\tilde{Y} \right)^2 \int_{2 \log_+ \tilde{Y}}^{\infty} e^{-x} (x+1) U(\mathrm{d}x) + \tilde{Y} \int_0^{2 \log_+ \tilde{Y}} (x+1) U(\mathrm{d}x) \right]$$

$$(3.14) \quad \leq c_{18} \mathbf{E} \left[\tilde{Y} \left(1 + \log_+ \tilde{Y} \right)^2 \right] = c_{18} \mathbf{E} \left[Y \left(1 + \log_+ Y \right)^2 \right]$$

On the other hand, in the same way, we obtain that

$$(3.15) \quad \mathbf{E}[\Sigma_2] \leq c_{19} \mathbf{E} \left[Z \left(1 + \log_+ Z \right) \right].$$

Consequently,

$$(3.16) \quad \mathbf{E}[\Sigma_1] + \mathbf{E}[\Sigma_2] \leq c_{20} \left(\mathbf{E}[Y + Z] + \mathbf{E} \left[Y \left(\log_+ Y \right)^2 \right] + \mathbf{E} \left[Z \log_+ Z \right] \right).$$

Note that (3.1) ensures that $\mathbf{E}[Y + Z] < \infty$. The (3.11) is thus proved and we completes the proof of (3.6). \square

3.2 (1.8) is a necessary condition

This subsection is devoted to proving that

$$(3.17) \quad \max \left\{ \mathbf{E} \left[Z \log_+ Z \right], \mathbf{E} \left[Y \left(\log_+ Y \right)^2 \right] \right\} = \infty \implies \mathbf{E}[D_{\infty}^{(0)}] = 0.$$

Proof of (3.17). According to (ii) of Theorem 3.1, we only need to show that

$$(3.18) \quad \forall y > 0, \mathbf{P}\text{-a.s.} \quad \sum_{n=1}^{\infty} \mathbf{E}_{\zeta_n} \left[X; R(\zeta_n) e^{-\zeta_n} X \geq y \right] = \infty.$$

We break the assumption on the left-hand side of (3.17) up into three cases. In each case, we find out a different lower bound for X to establish (3.18). It hence follows that $D_{\infty}^{(0)}$ is trivial as $\mathbf{E}[D_{\infty}^{(0)}] = 0$. The three cases are stated as follows:

$$(3.19a) \quad \mathbf{E}[Y(\log_+ Y)^2] = \infty, \quad \mathbf{E}[Y(\log_+ Y)] < \infty;$$

$$(3.19b) \quad \mathbf{E}[Y(\log_+ Y)] = \infty;$$

$$(3.19c) \quad \mathbf{E}[Z(\log_+ Z)] = \infty.$$

Proof of (3.18) under (3.19a) Recall that for any particle $x \in \mathbb{T} \setminus \{\emptyset\}$, $\Delta V(u) = V(u) - V(\overleftarrow{u})$, and that under \mathbf{P}_a , $(\Delta V(u); |u| = 1)$ is distributed as \mathcal{L} . For any $s \in \mathbb{R}$, we define a pair of random variables:

$$(3.20) \quad Y_+(s) := \sum_{|u|=1} e^{-\Delta V(u)} 1_{(\Delta V(u) > -s)}, \quad Y_-(s) := \sum_{|u|=1} e^{-\Delta V(u)} 1_{(\Delta V(u) \leq -s)}.$$

Clearly, $\tilde{Y} = Y_+(s) + Y_-(s)$.

It follows from (3.9) and (2.4) that under \mathbf{P}_{ζ_n} ,

$$\begin{aligned} X &\geq \frac{\sum_{|u|=1} c_1(1 + \zeta_n + \Delta V(u)) e^{-\Delta V(u)} 1_{(\Delta V(u) > -\zeta_n/2)}}{c_2(1 + \zeta_n)}, \\ &\geq \frac{\sum_{|u|=1} c_1(1/2 + \zeta_n/2) e^{-\Delta V(u)} 1_{(\Delta V(u) > -\zeta_n/2)}}{c_2(1 + \zeta_n)} \geq c_{21} Y_+(\zeta_n/2), \end{aligned}$$

where $\{(Y_+(s), Y_-(s)); s \in \mathbb{R}\}$ is independent of ζ_n and $c_{21} := \frac{c_1}{2c_2} > 0$. We thus see that the assertion that for any $y > 0$,

$$(3.21) \quad \sum_{n=1}^{\infty} \mathbf{E} \left[Y_+(\zeta_n/2); R(\zeta_n/2) e^{-\zeta_n} Y_+(\zeta_n/2) \geq y \middle| \zeta_n \right] = \infty, \quad \mathbf{P}\text{-a.s.},$$

yields (3.18). It is known that $\zeta_n \rightarrow \infty$ as n goes to infinity (see, for example, [4]). It suffices that

$$(3.22) \quad \sum_{n=1}^{\infty} F(\zeta_n/2, \zeta_n) = \infty, \quad \mathbf{P}\text{-a.s.}$$

where

$$(3.23) \quad F(s, z) := \mathbf{E} \left[Y_+(s); \log Y_+(s) \geq z \right], \quad s, z \in \mathbb{R}.$$

Let $F_1(z) := \mathbf{E}[Y; \log Y \geq z]$ which is positive and non-increasing. It follows from Lemma 2.1 and (3.1) that $\mathbf{E}[Y] = 1$. Therefore, for any $s, z \in \mathbb{R}$,

$$(3.24) \quad 0 \leq F(s, z) \leq F_1(z) \leq \mathbf{E}[Y] = 1.$$

On the one hand, we deduce from (3.19a) that

$$\begin{aligned} \int_0^{\infty} F_1(y) y dy &= \int_0^{\infty} \mathbf{E} \left[Y 1_{(\log Y \geq y)} \right] y dy = \mathbf{E} \left[Y \int_0^{(\log_+ Y)} y dy; Y \geq 1 \right] \\ &= \mathbf{E} \left[Y (\log_+ Y)^2 \right] / 2 = \infty. \end{aligned}$$

According to Proposition 2.6, \mathbf{P} -almost surely,

$$(3.25) \quad \sum_{n=1}^{\infty} F_1(\zeta_n) = \infty.$$

On the other hand, we can prove that $\sum_{n=1}^{\infty} [F_1(\zeta_n) - F(\zeta_n/2, \zeta_n)] < \infty$, \mathbf{P} -a.s. In fact, as $Y = Y_+(s) + Y_-(s)$ under \mathbf{P} , for any $s, y \in \mathbb{R}$,

$$\begin{aligned} F_1(y) - F(s, y) &= \mathbf{E} \left[Y 1_{(\log Y \geq y)} - Y_+(s) 1_{(\log Y_+(s) \geq y)} \right] \\ &= \mathbf{E} \left[Y 1_{(\log Y \geq y > \log Y_+(s))} + Y 1_{(\log Y_+(s) \geq y)} - Y_+(s) 1_{(\log Y_+(s) \geq y)} \right] \\ &= \mathbf{E} \left[Y 1_{(\log Y \geq y > \log Y_+(s))} + Y_-(s) 1_{(\log Y_+(s) \geq y)} \right]. \end{aligned}$$

Note that $Y \leq 2 \max\{Y_+(s), Y_-(s)\}$ under \mathbf{P} . It follows that

$$\begin{aligned} F_1(y) - F(s, y) &\leq \mathbf{E} \left[2Y_-(s) 1_{(\log Y \geq y > \log Y_+(s), Y_+(s) \leq Y_-(s))} + Y 1_{(\log Y \geq y > \log Y_+(s), Y_+(s) > Y_-(s))} \right] \\ &\quad + \mathbf{E} \left[Y_-(s) 1_{(\log Y_-(s) \geq y)} \right] \\ &\leq 3\mathbf{E} \left[Y_-(s) \right] + \mathbf{E} \left[Y 1_{(\log Y \geq y > \log Y_+(s), Y_+(s) > Y_-(s))} \right] \\ &\leq 3\mathbf{E} \left[Y_-(s) \right] + \mathbf{E} \left[Y 1_{(\log Y \geq y > \log(Y/2))} \right] =: d_1(s) + d_2(y). \end{aligned}$$

As a consequence,

$$(3.26) \quad \sum_{n=1}^{\infty} [F_1(\zeta_n) - F(\zeta_n/2, \zeta_n)] \leq \sum_{n \geq 0} d_1(\zeta_n/2) + \sum_{n \geq 0} d_2(\zeta_n).$$

Taking expectation on both sides yields that

$$\begin{aligned} \mathbf{E} \left[\sum_{n=1}^{\infty} (F_1(\zeta_n) - F(\zeta_n/2, \zeta_n)) \right] &\leq \mathbf{E} \left[\sum_{n \geq 0} d_1(\zeta_n/2) \right] + \mathbf{E} \left[\sum_{n \geq 0} d_2(\zeta_n) \right] \\ (3.27) \quad &= \int_0^{\infty} d_1(x/2) R(x) U(\mathrm{d}x) + \int_0^{\infty} d_2(x) R(x) U(\mathrm{d}x), \end{aligned}$$

where the last equality comes from (2.10).

For the first integration, we deduce from Lemma 2.1 that

$$(3.28) \quad d_1(s) = 3\mathbf{E} \left[Y_-(s) \right] = 3\mathbf{E} \left[\sum_{|x|=1} e^{-V(x)} 1_{(V(x) \leq -s)} \right] = 3\mathbf{P}(-S_1 \geq s).$$

By (2.4), (2.11) and (3.1),

$$\begin{aligned}
\int_0^\infty d_1(x/2)R(x)U(dx) &= 3 \int_0^\infty \mathbf{P}(-2S_1 \geq x)R(x)U(dx) \\
&= 3\mathbf{E}\left[\int_0^{-2S_1} R(x)U(dx); -2S_1 \geq 0\right] \\
&\leq c_{22}\mathbf{E}\left[\left(1 + (-2S_1)_+\right)^2\right] < \infty.
\end{aligned}$$

For the second integration on the right-hand side of (3.27), as $d_2(y) = \mathbf{E}\left[Y1_{(\log Y \geq y > \log(Y/2))}\right]$, we use (2.4), (2.11) and (3.19a) to obtain that

$$\begin{aligned}
\int_0^\infty d_2(x)R(x)U(dx) &= \int_0^\infty \mathbf{E}\left[Y1_{(\log Y \geq x > \log(Y/2))}\right]R(x)U(dx) \\
&= \mathbf{E}\left[Y \int_{(\log Y - \log 2)_+}^{\log_+ Y} R(x)U(dx)\right] \\
&\leq c_{23}\mathbf{E}\left[Y(1 + \log_+ Y)\right] < \infty.
\end{aligned}$$

Going back to (3.27), we conclude that

$$(3.29) \quad \mathbf{E}\left[\sum_{n=1}^\infty (F_1(\zeta_n) - F(\zeta_n/2, \zeta_n))\right] \leq \mathbf{E}\left[\sum_{n \geq 0} d_1(\zeta_n/2)\right] + \mathbf{E}\left[\sum_{n \geq 0} d_2(\zeta_n)\right] < \infty.$$

Therefore, \mathbf{P} -a.s.,

$$(3.30) \quad \sum_{n=1}^\infty [F_1(\zeta_n) - F(\zeta_n/2, \zeta_n)] < \infty,$$

which, combined with (3.25), implies (3.22). Thus (3.18) is proved under (3.19a).

Proof of (3.18) under (3.19b) Now we suppose that $\mathbf{E}[Y \log_+ Y] = \infty$. By (2.4), we observe that under \mathbf{P}_{ζ_n} ,

$$\begin{aligned}
X &= \frac{\sum_{|u|=1} R(\Delta V(u) + \zeta_n) e^{-\Delta V(u)} 1_{(\Delta V(u) > -\zeta_n)}}{R(\zeta_n)} \\
(3.31) \quad &\geq c_1 \frac{Y_+(\zeta_n)}{R(\zeta_n)},
\end{aligned}$$

where $\{Y_+(s); s \in \mathbb{R}\}$ and ζ_n are independent.

To establish (3.18), we only need to show that for any $y \geq 1$,

$$(3.32) \quad \sum_{n \geq 1} \mathbf{E}\left[\frac{Y_+(\zeta_n)}{R(\zeta_n)}; Y_+(\zeta_n) \geq ye^{\zeta_n} \middle| \zeta_n\right] = \sum_{n \geq 1} \frac{F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)} = \infty, \quad \mathbf{P}\text{-a.s.}$$

For any $y \geq 1$ fixed, let

$$(3.33) \quad F_2(x) := \frac{F_1(\log y + x)}{R(x)}, \quad \forall x \geq 0,$$

which is non-increasing as $R(x) = U^-([0, x])$ is non-decreasing and F_1 is non-increasing. One sees that

$$(3.34) \quad \sum_{n \geq 1} F_2(\zeta_n) = \sum_{n \geq 1} \frac{F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)} + \sum_{n \geq 1} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)}.$$

By (2.4), $\frac{F_1(\log y + x)}{c_2(1+x)} \leq F_2(x) \leq \frac{1}{c_1}$. It then follows from (3.19b) that

$$\begin{aligned} \int_0^\infty F_2(x) x \, dx &\geq \int_0^\infty F_1(\log y + x) \frac{x}{c_2(1+x)} \, dx \\ &\geq \int_1^\infty c_{24} \mathbf{E} \left[Y 1_{(\log Y \geq \log y + x)} \right] \, dx \\ &\geq c_{24} \mathbf{E} [Y (\log Y - \log y - 1)_+] = \infty. \end{aligned}$$

By Proposition 2.6,

$$(3.35) \quad \sum_{n \geq 0} F_2(\zeta_n) = \sum_{n \geq 0} \frac{F_1(\log y + \zeta_n)}{R(\zeta_n)} = \infty, \quad \mathbf{P}\text{-a.s.}$$

In view of (3.34) and (3.35), it suffices to show that \mathbf{P} -a.s.,

$$(3.36) \quad \sum_{n \geq 0} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)} < \infty.$$

Recall that $F_1(z) - F(s, z) \leq d_1(s) + d_2(z)$. By (2.10),

$$\begin{aligned} (3.37) \quad &\mathbf{E} \left[\sum_{n \geq 0} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)} \right] \\ &\leq \mathbf{E} \left[\sum_{n \geq 0} \frac{d_1(\zeta_n) + d_2(\log y + \zeta_n)}{R(\zeta_n)} \right] = \int_0^\infty \left[d_1(x) + d_2(\log y + x) \right] U(dx). \end{aligned}$$

On the one hand, recalling that $d_1(x) = 3\mathbf{P}(-S_1 \geq x)$, we deduce from (2.11) that

$$\begin{aligned} (3.38) \quad \int_0^\infty d_1(x) U(dx) &= \int_0^\infty 3\mathbf{P}(-S_1 \geq x) U(dx) \\ &= 3\mathbf{E} \left[\int_0^{(-S_1)_+} U(dx) \right] \\ &\leq 3c_4 \mathbf{E} [1 + (-S_1)_+] < \infty. \end{aligned}$$

On the other hand, recalling that $d_2(x) = \mathbf{E}[Y; \log Y \geq x > \log Y - \log 2]$, by (2.11) again, we obtain that

$$\begin{aligned}
(3.39) \quad \int_0^\infty d_2(\log y + x)U(dx) &= \int_0^\infty \mathbf{E}[Y 1_{(\log Y \geq \log y + x > \log Y - \log 2)}]U(dx) \\
&= \mathbf{E}\left[Y \int_{(\log Y - \log y - \log 2)_+}^{(\log Y - \log y)_+} U(dx)\right] \\
&\leq c_4(1 + \log 2)\mathbf{E}[Y] < \infty.
\end{aligned}$$

Combined with (3.38) and (3.39), (3.37) becomes that

$$(3.40) \quad \mathbf{E}\left[\sum_{n \geq 1} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)}\right] < \infty.$$

We thus get (3.36), and completes the proof of (3.18) given (3.19b).

Proof of (3.18) under (3.19c) In this part we assume that $\mathbf{E}[Z \log_+ Z] = \infty$ with $Z = \sum_{|u|=1} V(u)_+ e^{-V(x)} \geq 0$. We observe that under \mathbf{P}_{ζ_n} ,

$$\begin{aligned}
(3.41) \quad X &\geq \frac{\sum_{|u|=1} R(\Delta V(u) + \zeta_n) e^{-\Delta V(u)} 1_{(\Delta V(u) > 0)}}{R(\zeta_n)} \\
&\geq \frac{c_1}{R(\zeta_n)} \tilde{Z},
\end{aligned}$$

where $\tilde{Z} = \sum_{|x|=1} (\Delta V(x))_+ e^{-\Delta V(x)}$ is independent of ζ_n . As a consequence, for any $y > 0$,

$$(3.42) \quad \sum_{n \geq 1} \mathbf{E}_{\zeta_n} \left[X; R(\zeta_n) e^{-\zeta_n} X \geq y \right] \geq \sum_{n \geq 1} \frac{c_1}{R(\zeta_n)} \mathbf{E} \left[\tilde{Z}; c_1 \tilde{Z} \geq y e^{\zeta_n} \middle| \zeta_n \right].$$

Recall that \tilde{Z} is distributed as Z under \mathbf{P} . Therefore, it is sufficient to prove that for any $y > 0$,

$$(3.43) \quad \sum_{n \geq 1} \frac{1}{R(\zeta_n)} \mathbf{E} \left[\tilde{Z}; \tilde{Z} \geq y e^{\zeta_n} \middle| \zeta_n \right] = \sum_{n \geq 1} F_3(\zeta_n) = \infty, \quad \mathbf{P}\text{-a.s.}$$

where

$$(3.44) \quad F_3(z) := \frac{\mathbf{E}[Z; \log Z \geq z + \log y]}{R(z)}, \quad \forall z \geq 0.$$

Since R is non-decreasing, the function F_3 is non-increasing. By Lemma 2.1 and (2.4),

$$(3.45) \quad 0 \leq F_3(z) \leq \frac{\mathbf{E}[Z]}{R(z)} \leq \frac{\mathbf{E}[(S_1)_+]}{c_1} < \infty.$$

Moreover, by (2.11) and (3.19c),

$$\begin{aligned}
 (3.46) \quad \int_0^\infty F_3(x)x \, dx &\geq \int_1^\infty c_{25} \mathbf{E} \left[Z; \log Z - \log y \geq x \right] dx \\
 &\geq c_{25} \mathbf{E} \left[Z(\log Z - \log y - 1)_+ \right] = \infty.
 \end{aligned}$$

Because of Proposition 2.6, we obtain that for any $y > 0$,

$$(3.47) \quad \sum_{n \geq 1} \frac{1}{R(\zeta_n)} \mathbf{E} \left[\tilde{Z}; \tilde{Z} \geq ye^{\zeta_n} \middle| \zeta_n \right] = \sum_{n \geq 1} F_3(\zeta_n) = \infty, \quad \mathbf{P}\text{-a.s.}$$

which completes the proof of (3.18) under (3.19c).

□

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